DIMENSIONAL ANALYSIS USING TORIC IDEALS

M. A. ATHERTON, R. A. BATES, AND H. P. WYNN

Abstract. Classical dimensional analysis is one of the cornerstones of qualitative physics and is also used in the analysis of engineering systems, for example in engineering design. The basic power product relationship in dimensional analysis is identical to one way of defining toric ideals in algebraic geometry, a large and growing field. This paper exploits the toric representation to provide a method for automatic dimensional analysis for engineering systems. In particular all "primitive", invariants for a particular problem, in a well defined sense, can be found using such methods.

1. Dimensional analysis

Dimensional analysis has a long history. It was discussed by Newton and provided useful intuition to Maxwell, see [9], chapter 3. A recent paper giving a pleasant popular overview is [3]. The first rigorous and most well-known treatment is by Buckingham [4], whose name is attached to the main theorem. Dimensional analysis is still considered a fundamental part of physics and is taught at an early stage in schools and colleges as a basic part of the physics syllabus. It is often covered under a heading of qualitative physics [2]. In engineering it gives a useful additional tool for the analysis of systems [11]. It is used in engineering design and in the formal design of engineering experiments [10] [12]. It has also been used in economics [1]. For an interesting recent application to turbulence and criticality see [5] [6].

We shall give an algebraic development of dimensional analysis based on the theory of toric ideals and toric varieties. Although this is essentially a reformulation, the algebraic theory itself is by no means elementary. The theory of toric ideals is a live branch of algebraic geometry. We have used [16] and the recent comprehensive volume [8]. We shall see that the methods give all "primitive" invariants for a particular problem, in a well-defined sense.

Within mathematical physics dimensional analysis can also be seen as an elementary application of the theory of Lie groups and invariants, when the group is the scale group defined by multiplication. We shall draw on [14] in the penultimate section.

The basic idea of dimensional analysis is that physical systems use fundamental quantities, or units, of mass (M), length (L) and time (T). To this list may been added various others such as temperature (K) and current (I), depending on the domain. The extent to which new fundamental quantities can be expressed in terms of M, L, T goes to the heart of physics but we shall not delve deeply. Mathematical models for physical systems use so-called *derived* quantities such as: force, energy, momentum, capacity etc. Dimensional analysis tells us that each one of these quantities has units which have a power product representation. Table 1 gives a few examples from mechanics.

Quantity	units
momentum	$MLT^{-}1$
force	MLT^{-2}
work	ML^2T^{-2}
energy	ML^2T^{-2}
pressure	$ML^{-1}T^{-2}$
density	ML^{-3}
volumetric flow	L^3T^{-1}

Table 1. Some basic derived quantities

We note that the formulae for the expression of derived units have integer powers. This is critical for our development: it makes them *algebraic* in the sense of polynomial algebra.

In a physical system we may be interested in a special collection of derived quantities. The task of dimensional analysis is to derive dimensionless variables with a view to finding, by additional theory or experiment, or by both, the relationship between these dimensionless quantities. As mentioned, the key theorem in the area is due to Buckingham. In this section we explain it with an example, leaving a more detailed discussion until later.

Rather than use the $M, L, T \dots$ notation we assume that there are some basic quantities of interest which we label $z_1, z_2, \dots$. Each quantity is assumed to have the scaling property, namely if the fundamental units, which we now call $t_1, t_2, \dots$ are scaled up or down this induces a transformation on the z_i. Whether this means simply a change in units or actual physical scaling of the system is sometimes unclear in the literature, but we shall prefer the latter interpretation.

As example, if z_1 is force and the fundamental units are mass (t_1), length (t_2) and time (t_3), then the scaling transformation is

$$z_1 \to t_1 t_2 t_3^{-2} z_1.$$

With a collection of derived quantities we have one such transformation for each z_j. A slightly more realistic formulation is to introduce non-zero constants c_j so that in this case we would have

$$z_1 \to c_1 t_1 t_2 t_3^{-2} z_1,$$

but this would make little difference to our derivations

Here is a well-known example which we shall use as our running example. It concerns a body in a fluid and the quantities of interest are fluid density (z_1), fluid velocity (z_2), object diameter (z_3), fluid viscosity (z_4)and fluid resistance (z_5). Taking the units into account the transformation is:

$$\begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \end{pmatrix} \to \begin{pmatrix} t_1 t_3^{-3} z_1 \\ t_2^{-1} t_1 z_2 \\ t_3 z_3 \\ t_1 t_2^{-1} t_3^{-1} z_4 \\ t_1 t_2^{-2} t_3 z_5 \end{pmatrix} \tag{1.1}$$

After a little algebra, or formal use of Buckingham's theorem, we can derive dimensionaless quantities

$$y_1 = z_1 z_2 z_3 z_4^{-1}, \;\; y_2 = z_1^{-1} z_2^{-2} z_3^{-2} z_5.$$

The first quantity is Reynolds number. The term *dimensionless* is interpreted by saying that replacing each z_j by the y_j in the transformation $\rightarrow$ in 1.1, leaves the expression unchanged: the y_j are rational invariants of the transformation. The dimensionless principal, for our example, embodied in the Buckingham theorem is that *any* function F of $x_1, \dots, x_5$ which is invariant under $\rightarrow$ is a function of y_1 and y_2 which we write: $F(y_1, y_2)$.

We now sketch the traditional method. The transformation $\rightarrow$ can be coded up by capturing the exponents in the power products. This gives

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & -1 & 0 & -1 & -2 \\ -3 & 1 & 1 & -1 & 1 \end{pmatrix}.$$

This matrix has rank 3 and we can find a full rank 2×5 kernel matrix K. Namely, a K which has rank 2 such that $A^T K = 0$. This is readily computed using existing functions in computer algebra such as the "nullspace" command on Maple. We obtained

$$K = \begin{pmatrix} 1 & 1 & 1 & -1 & 0 \\ -1 & -2 & -2 & -2 & 1 \end{pmatrix}.$$

The key point is that the rows of this K give the exponents of $z_1, \dots, z_5$ in y_1 and y_2. However, we can also derive alternative K. For example,

$$K' = \begin{pmatrix} 1 & 1 & 1 & -1 & 1 \\ 0 & -1 & -1 & -1 & 1 \end{pmatrix}.$$

This gives an alternative to y_2, above, namely $y_3 = t_2^{-1} t_3^{-1} t_4^{-1} t_5$. The toric approach clarifies, among other issues, the immediate problem of the choice of K which this example exposes.

2. Toric ideals

Algebraic geometry is concerned with *ideals* and their counterpart algebraic *varieties*. We give a very short description here. (Note that we shall use x for variables in an abstract algebraic setting reserving z for "real" problems.) A standard reference is [8]. We start with the ring of all polynomials in n variables $\{x_1, \dots, x_n\}$ over a field k: $k[x_1, \dots, x_n]$. A set I of polynomials is an ideal if $F \in I$ implies $s(x)f(x)$ is in I for any $s(x)$ in $k[x_1, \dots, x_n]$. By a theorem of Hilbert all ideals are finitely generated. That is we can find a set of polynomials $f_1(x), \dots f_m(x)$ such that any $f(x)$ in $k[x_1, \dots, x_n]$ can be written $f(x) = s_1(x)f_1(x) + \dots + s_m(x)f_m(x)$ for some $\{s_i(x)\}$ in $k[x_1, \dots, x_n]$. An ideal I gives a variety as the set of x such that $f(x) = 0$ for all $f(x) \in I$. The other identity that I is the set of all polynomials zero on the variety is not always true, but for the purposes of this paper we will alternate freely between varieties and ideal. It will also be enough to work within the field Q of rationals.

Modern computational algebra has benefitted hugely from the theory of Gröbner bases and the algorithms that grew out of the theory, notably the Buchburger algorithm. We will need one more concept, that of a monomial term ordering, or term ordering for short. Monomials $x^\alpha = x_1^{\alpha_1} ... x_n^{\alpha_n}$, where $\alpha = \alpha_1, \dots, \alpha_n \geq 0$ ie $\alpha_i \geq 0,\ i = 1, \dots, n$, drive the theory. A monomial term ordering, written $x^\alpha \prec x^\beta$ between is a total (linear) ordering with the addition condition: $x^\alpha \prec x^\beta$ implies $x^{\alpha+\gamma} \prec x^{\beta+\gamma}$, for all $\gamma \geq 0$. Since such an ordering is linear every polynomial f has a leading term $LT_\prec(f)$. If we fix the monomial ordering, $\prec$, the Gröbner basis

$G_\prec = \{g_1(x), \ldots, g_m(x)\}$ of an ideal I with respect to $\prec$ is a basis such that the ideal generated by *all* monomials in the ideal is the same as that generate by the leading terms of $G_\prec$. Given I and $\prec$ the Buchburger algorithm delivers $G_\prec$. We will be concerned with the set of all Gröbner bases as $\prec$ ranges over all monomial term orderings. This is called the *fan* and is finite, although is can be very large.

One of the main definitions of a toric ideal fits perfectly with the power product transformations of dimensional analysis. It is this observation which motivates this paper. We will emphasize the connection by using the same notation: $\{t, y, A\}$, with x or z according to emphasis, but with t, y and A used in both the pure algebra and physical theories.

The following development can be taken from an number of books, but [16] is our main source. The main steps in the definition are.

(1) The polynomial ring over n variables $k[\mathbf{x}] = k[x_1, \ldots, x_n]$.
(2) A $d \times n$ matrix A with columns labeled $\mathbf{a_1}, \ldots, \mathbf{a_d}$.
(3) Variables $t_1, \ldots, t_d$ and the Laurent ring generated by the t_i and the inverses t_i^{-1}. We write this as

$$k[\mathbf{t}, \mathbf{t}^{-1}] = k[t_1, \ldots, t_d, t_1^{-1}, \ldots t_d^{-1}].$$

(4) A power product mapping from $k[\mathbf{x}]$ to $k[\mathbf{t}, \mathbf{t}^{-1}]$ defined by A:

$$x_i \to t^{\mathbf{a}_i}$$

The kernel of the mapping in (4) above is the toric ideal. It can be considered as the ideal obtained by formally eliminating $\mathbf{t}$ t from the ideal:

$$\langle x_i - t^{\mathbf{a}_i}, i = 1, \ldots, n\rangle$$

The following paragraph should be considered as a theorem.

The generators of the toric ideal ideal are related to the kernel of A in the follow way. The generators are all so-called *binomials*

$$\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}},$$

where $\mathbf{u}$ and $\mathbf{v}$ are non-negative integer vectors with the property that

$$A\mathbf{u} = A\mathbf{v}.$$

The last equation can be written $A(\mathbf{u} - \mathbf{v}) = \mathbf{0}$, which is equivalent to $\mathbf{u} - \mathbf{v}$ being in the kernel of A.

The connection with dimensional analysis should now be clear. Let us put dimensional analysis on a similar notational footing, only using z instead of x. Start with a $d \times n$ matrix A with columns $\{\mathbf{a}_i\}$. The general form of the mapping $\to$ in 1.1 becomes

$$z_i \to \mathbf{t}^{\mathbf{a}_i} z_i,\ i = 1, \ldots, n \tag{2.1}$$

We can write this in matrix terms as

$$\mathbf{z} \to \mathbf{t}^{\mathbf{A}}\mathbf{z} \tag{2.2}$$

Now, suppose we have a possible invariant y_j. Using $\mathbf{u}, \mathbf{v}$ to denote integer vectors with non-negative entries to distinguish the positive from the negative exponents and write

$$y_j = \mathbf{z}^{\mathbf{u_j}}\mathbf{z}^{-\mathbf{v_j}}$$

The condition to be an invariant is that substituting each z_j by y_j in the right hand side of 2.2 for z leaves y_j unchanged. But the condition for this is

$$\mathbf{z}^{\mathbf{u}}\mathbf{z}^{-\mathbf{v}} = (\mathbf{t}^{A}\mathbf{z})^{\mathbf{u}}(\mathbf{t}^{A}\mathbf{z})^{-\mathbf{v}}, \quad j = 1, \ldots, d,$$

which is equivalent to

$$A\mathbf{u}_j - A\mathbf{v}_j = 0, \quad j = 1, \ldots, d,$$

exactly the toric condition. We have proved our main result:

Theorem 2.1. *A variable y is a dimensional invariant in a system defined by a matrix A, with derived variable $\mathbf{z}$, if any only if it takes the form*

$$y = \mathbf{z}^{\mathbf{u}}\mathbf{z}^{-\mathbf{v}}$$

where u and v are non negative integer vectors such that $A\mathbf{u} = A\mathbf{v}$. Moreover the set of all quantities

$$\mathbf{z}^{\mathbf{u}} - \mathbf{z}^{\mathbf{v}},$$

is the toric ideal I_A with generator matrix A.

A brief summary is to say that the set of all dimensional quantities y associated with A are exactly those given by the toric ideal I_A.

We can give a minimal set of generators for the toric ideal of our running example. We use the "Toric" function on the computer algebra package CoCoa [7], which takes the matrix A as input. Simply to ease the notation in the use of computer algebra we use $a, \ldots, e$, for $z_1, \ldots, z_5$. The script with output is.

Use $R ::= QQ[a, b, c, d, e]$;
Toric([[1,0,0,1,1],[-3,1,1,-1,1],[0,-1,0,-1,-2]]);
Ideal$(-d^2 + ae, abc - d, bcd - e)$

By the theorem, given any generator we have a invariant. Thus $-d^2 + ae$ yields $\frac{ae}{d^2}$. Thus we have gives three invariants:

$$\frac{ae}{d^2}, \ \frac{abc}{d}, \ \frac{bcd}{e}$$

We see that the second two ideal generators give exactly the dimensional variables from the kernel matrix K', above. A key point is that the toric ideal may have more generators than the rank of the kernel in Buckingham's theorem. The next section explains why this is so.

2.1. **Saturation and Gröbner bases.** To summarise, the toric version of dimensional analysis says that we can generate dimensionless quantities from the toric ideal which is the elimination ideal of the original power product representation, being careful to use elimination in the proper algebraic sense.

A lattice ideal associated with an integer defining matrix A is the ideal based on a full rank kernel matrix. That is if A is $d \times n$ with rank d then we find an integer $n \times d - n$ matrix K, with rank $n - d$ with rows $\mathbf{k}_1, \ldots, \mathbf{k}_{n-d}$ with $A^T K = 0$.

The corresponding lattice ideal is generated by $\{\mathbf{t}^{\mathbf{k}_j}\}$. For our first K in 1 *lattice ideal* has two generators:

$$\langle z_1 z_2 z_3 z_4^{-1}, z_1^{-1} z_2^{-2} z_3^{-2} z_4^{-2} z_5 \rangle.$$

But, as we have seen, this has one fewer generators than the toric ideal. However, given any such lattice ideal we can obtain the toric ideal using a process called

saturation. The process has two steps. Fix the defining matrix A and let I_A be a lattice ideal associated with A.

(1) Select a dummy variable s and adjoin to the lattice ideal the generator $s\prod_{j=1}^n x_j + 1$. That is form the union
$$I_K^* = I_K \cup \langle s\prod_{j=1}^n x_j + 1\rangle.$$
(2) Eliminate s from I_K^* to give the toric ideal for $\{x_1, \ldots, x_n\}$. That is, the toric ideal is obtained as the elimination ideal for $\{x_1, \ldots, x_n\}$.

The process of elimination in this saturation process is a formal procedure and leads to a *reduced Gröbner basis* of the toric ideal which in general depends in general on the monomial ordering used in the elimination algorithm.

This process gives an explanation for the fact that the toric ideal contains, but is not necessarily equal to the lattice ideal. Recall that unions of ideals is mirrored by intersections of varieties. The addition condition $s\prod_{j=1}^n x_j + 1 = 0$ giving the variety defined by I_A^* forces all the x_j to be nonzero. This property is inherited by the toric ideal. It implies that if any $x_j = 0$ then all x_j and zero. That is to say, saturation removes the principal axes and all axial subspaces.

This gives a nice physical interpretation. If we exclude the origin, then for the toric variety associated with the toric ideal *must not contain any other zeros.* Translated into the original z_j variables, the toric ideal description of the dimensionless quantities is appropriate when *non of the defining variables z_j is allowed to be zero.* This removal of zeros is intimately connected with the abstract definitions of toric varieties based on the concept of a torus in complex variables, but we do not develop this here, see [8].

2.2. **The Gröbner fan, primitive invariants and the Graver basis.** A natural question given the ease of computing invariants using toric methods is whether the invariants obtained in this way are in some sense minimal. This turns out to be the case. We can illustrate this with our example. A little inspection of the basis $-d^2 + ae, abc - d, bcd - e$ shows that we cannot get simpler invariants from this basis by multiplication (or division) : eg if
$$y_1 = \frac{ae}{d^2},\ y_3 = \frac{bcd}{e}$$
then $y_1 y_2 = \frac{abc}{d}$ which, although a new invariant, is not obtained by reducing the numerator or denominator of any of the original invariants.

Definition 2.2. A basis element $z^u - z^v$ of I_A is called is called *primitive* if there is no basis element invariant $z^{u'} - z^{v'}$ such that such that $z^{u'}$ divides z^u and $z^{v'}$ divides z^v. We call an invariant $y = z^u z^{-v}$ *primitive* if and only if $z^u - z^v$ is primitive as a basis element of I_A.

Lemma 4.6 of [16] is

Theorem 2.3. *Every invariant obtained from as a reduced Gröbner basis of I_A is primitive.*

Note that in what follows we are a little lazy in not disassociating an invariant from its inverse.

As mentioned, as we range over all monomial term orderings defining the individual Gröbner basis we obtain the complete Gröbner fan and by the lemma and our definition all resulting primitive invariants are primitive. This union of bases is called the *universal* Gröbner basis and the computer programme Gfan is recommended to compute the fan [13].

We return to our running example. If we put the G-basis element $\langle -d^2+ae, abc-d, bcd-e\rangle$ into Gfan we obtain the full fan as

$$\langle bcd-e, ae-d^2, abc-d\rangle,\ \langle e-bcd, abc-d\rangle,\ \langle d^2-ae, bcd-d, abc-d\rangle$$

$$\langle d-abc, ab^2c^2-e\rangle,\ \langle e-ab^2c^2, d-abc\rangle,$$

the first of which is the input basis.

The universal Gröbner basis of distinct basis terms (ignoring the sign change) copy is:

$$bcd-e, ae-d^2, abc-d, ab^2c^2-e$$

and we have a new primitive invariant: $\frac{ab^2c^2}{e}$.

The set of *all* primitive polynomials, which may be larger than that giving the union of the basis elements in the fan, is called the *Graver* basis. Algorithm 7.2 of [16] can be used for this.

Briefly, the method starts by constructing from A an extended matrix called the Lawrence lifting:

$$\tilde{A} = \begin{bmatrix} A & 0 \\ I & I \end{bmatrix},$$

where the zero is a $d \times n$ zero matrix and I is a $d \times d$ identity matrix. Then introducing n more derived variables to make a set $z_1, \ldots, z_n, z_{n+1}, \ldots z_{2n}$ a toric ideal is constructed using $\tilde{A}$. Finally, set $z_{n+1} = \cdots = z_{2n} = 1$

The method is conveniently set out in the help screen of “ToricIdealBasis” on Maple. After inputting

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & -2 & 0 & 0 & 0 & 0 & 0 \\ -3 & 1 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

we use the commands

$zs := [seq(z[i], i = 1..10)];$
$T := ToricIdealBasis(A, zs, plex(op(zs)), method = ' hs', grading = grd);$
$G := subs([seq(zs[i] = 1, i = 6..10)], T);$

This yields

$$\langle z_2z_3z_4 - z_5, z_1z_5 - z_4^2, -z_4 + z_1z_2z_3, -z_5 + z_1z_2^2z_3^2\rangle,$$

In this case the set is the same as given by the fan. That is, the universsal Gröbner basis and the Graver basis are the same.

3. Further examples

For each of the examples we give the derived quantities using the classical notation, (i) the A matrix (ii) a single toric ideal basis give by the default function on CoCoa and (iii) a full set of primitive basis elements, that is the Graver basis, given by the maple 'ToricIdealBasis" command. From this a full set of primitive invariants is immediate. It turns out that for all except one of our examples (windmill) the Graver basis is also the universal Gröbner basis. We try to mention when we find well-known invariants.

3.1. **Windmill.** This standard problem is taken from [11], Section 9.3. (We have changed d there to D).

It concerns a simple windmill widely used to pump water. The table is

shaft power, P	ML^2Y^{-2}
diameter, D	L
wind speed, V	LT^{-1}
rotational speed, n	T^{-1}
air density ρ	ML^{-3}

The A-matrix is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 2 & 1 & 1 & 0 & -3 \\ -3 & 0 & -1 & -1 & 0 \end{pmatrix}$$

In the $a, b \ldots$ notation we obtain, from Cocoa, a basis with 4 terms :

$$\langle bd - c, b^2c^3e - a, c^5e - ad^2, bc^4e - ad\rangle$$

The first entry give a dimensionless quantities discussed in the book:

$$\frac{V}{nD}.$$

The universal Gröbner basis obtained from the Gfan gives five terms

$$\langle bd - c, b^2c^3e - a, c^5e - ad^2, bc^4e - ad, b^5d^3e - a\rangle$$

The last of these is also discussed in the book; it gives the invariant

$$\frac{P}{\rho n^3 d^5}.$$

A full set of 7 primitive invariants, the Graver basis, is

$$\langle bd - c, b^2c^3e - a, c^5e - ad^2, bc^4e - ad, b^5d^3e - a, b^4cd^2e - a, bc^4e - ad\rangle,$$

Since $\mathrm{rank}(A) = 3$ there are only two algebraically independent invariants. The standard argument may suggest testing the relationship between any two independent invariants, for example in a wind tunnel. An important question, which should be the subject of further research, is say which two or, more generally, whether the dimensional analysis is sufficiently trusted to test only one pair and infer other relationships from the algebra.

3.2. **Forced convection.** The interest is in the following derived quantities: the forced convection coefficient h, the velocity, u, the characteristic length of the heat transfer surface L, the conductivity of the fluid k, the viscosity, μ, the fluid specific heat capacity, c and the fluid density, ρ. The fundamental dimensions are M, L, T and two new ones temperature (K) and energy (J). With columns in the order of the listed the rows in the units order the A-matrix is

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & -1 & 1 \\ -2 & 1 & 1 & -1 & -1 & 0 & -3 \\ -1 & -1 & 0 & -1 & -1 & 0 & 0 \\ -1 & 0 & 0 & -1 & 0 & -1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}.$$

Resorting to the $a, b, \ldots$ notation we have from, Cocoa, the ideal

$$\langle ac - d, ef - d, bcg - e, bfg - a, ae - bdg \rangle,$$

giving invariants:

$$\frac{ac}{d}, \frac{ef}{d}, \frac{bcg}{e}, \frac{bfg}{a}, \frac{ae}{bdg}$$

The first three of these are well-known invariants:

$$\text{Reynolds number} : R = \frac{\rho u L}{\mu} \tag{3.1}$$

$$\text{Nusselt number}, N = \frac{hL}{k} \tag{3.2}$$

$$\text{Prandtl number}, P = \frac{\mu c}{k} \tag{3.3}$$

In preparing this paper it was pleasing to obtain these directly from the computer on the first run. The full set of 7 primitive basis elements is

$$\langle ac - d, ef - d, bcg - e, bfg - a, ae - bdg, bcfg - d, ac - ef \rangle$$

The simplest of the "new" primitive invariants is from $ac - ef$:

$$\frac{hL}{\mu c},$$

which is the Reynolds/Nusselt.

3.3. **Electrodynamics.** As an exercise we take six basic quantities for electrodynamics, used the literature to give some expression in terms of mass (M), length (L), Time (L) and current (A). We do not have any particular elctromagnetic device in mind, but simply try to find some dimensionless quantities. The table below gives one version:

Quantity	units
charge	TA
potential	$ML^2T^{-3}A^{-1}$
capacitance	$M^{-1}L^{-2}T^4A^2$
inductance	$ML^2T^{-2}A^{-2}$
resistance	$ML^2T^{-3}A^{-2}$

The A-matrix is

$$A = \begin{pmatrix} 0 & 1 & -1 & 1 & 1 \\ 0 & 2 & -2 & 2 & 2 \\ 1 & -3 & 4 & -2 & -3 \\ 1 & -1 & 2 & -2 & -2 \end{pmatrix}.$$

CoCoa gives

$$\langle bc - a, -ce^2 + d, -ae^2 + bd \rangle.$$

Note that A only has rank 3. It turns out that this is a complete list of primitive basis elements.

3.4. **Quantum.** Toric ideals are embedded in advanced models in physics but one can get some way with simple dimensional analysis. This example is given in some form by a number of authors. We found [15], section 1.3.1, useful. The hydrogen atom consists of a proton and a neutron and the Bohr radius is the distance between them. We have used slightly non-standard notation. In a somewhat cavalier manner we have introduced the speed of light as derived quantity.

mass of electron, m_e	M
Bohr radius, a_0	L
energy, E	$ML^{-2}T^{-2}$
Plank's constant, $\hbar$	ML^2T^{-1}
permitivity of vacuum (squared), e^2	ML^3T^{-2}
speed of light, c	LT^{-1}

Then the A-matrix is

$$\begin{pmatrix} 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & 3 & 1 \\ 0 & 0 & -2 & -1 & -2 & -1 \end{pmatrix}$$

The ideal is

$$\langle -df + e, bc - df, abe - d^2, -cd^2 + ae^2, abf - d, af^2 - c, aef - cd \rangle,$$

(the algebraic e is e^2 and the algebraic c should not be confused with the speed of light). The first terms gives an invariant called the "fine structure constant"

$$\frac{e^2}{\hbar c}.$$

If we take the third term and interpret $\frac{x^2}{uvy}$ being invariant as stating that $v =$ constant $\times \frac{x^2}{uy}$, then we have a well known formula for a_0 interpreted as the size of the hydrogen atom:

$$a_0 = \text{constant} \times \frac{\hbar^2}{m_e e^2}.$$

We cannot resist stating that the sixth basis element, $af^2 - c$ gives

$$E = \text{constant} \times m_e c^2.$$

The Graver basis gives a full set of 10 primitive invariants for the hydrogen atom is

$$\langle -df+e, bc-df, abe-d^2, -cd^2+ae^2, abf-d, af^2-c, aef-cd, bc-e, abf^2-e, -f^2+ab^2c \rangle$$

It is not known whether this list has been given explicitly before.

4. Group invariance

Dimensional analysis should be considered as a special case of the theory of groups invariance and in an attempt to suggest a natural generalisation we very briefly sketch the theory of invariants.

We start with the action of a Lie group G acting on a manifold M in R^d. The manifold will be our model and the group something to do with our physical understanding of the physics being modelled. The *orbit* of $\mathcal{O}(x)$ is a point x in M be the set of all $g(x)$ for all g in G. If M is invariant under G then $\mathcal{O}(x) \subset M$. This sets up an equivalence relations with members of M in the same orbit being equivalent. The collection of equivalence classes is denoted by the quotient M/G and the *projection* $\pi : M \to M/G$ maps every member of of M into its correct equivalence class. If we are lucky then M/G is a manifold in its own right and we say that G acts *regularly* on M. Also, the mapping π can be used to set up a coordinate system on M/G and note that π itself is an invariant. This discussion leads naturally to the following

Proposition 4.1. *Let a group G act regularly on a manifold M. Consider a manifold defined by a smooth function F is a set $S_F = \{x|f(x) = 0$. Its is G-invariant is and only there is a function F^* defining a smooth sub-manifold $S_{F^*} = \{y|F^*(y) = 0\}$ on M/G such that*

$$S_{F^*} = \pi(S_F),$$

where π is the projection form M to M/G.

A one parameter Lie group G shifts a point x along an integral curve $\Psi(\epsilon, x)$ called a *flow* If we expand $\Psi(\epsilon, x)$ in a Taylor expansion in ϵ we obtain:

$$\Psi(\epsilon) = x + \epsilon\xi(x) + O(\epsilon^2).$$

The term $\xi(x) = (\xi_1(x), \ldots, \xi_d(x)$ defines a vector field and we can write v in local coordinates in classical

$$v = \xi_1(x)\frac{\partial}{\partial x_1} + \cdots + \xi_1(x)\frac{\partial}{\partial x_d}$$

A function ψ is an invariant if $v\psi = 0$ or

$$\xi_1(x)\frac{\partial \psi}{\partial x_1} + \cdots + \xi_1(x)\frac{\partial \psi}{\partial x_d} = 0.$$

This is a first order partial differential equation which can be solved by writing down

$$\frac{dx_1}{\xi_1(x)} = \cdots = \frac{dx_1}{\xi_d(x)},$$

namely by the methods of characteristics. The solutions take the form:

$$\psi_1(x) = c_1, \cdots, \psi_m(x) = c_m,$$

where the ψ_j are the invariants.

In our notation ϵ becomes t and the mapping $\to$ in (1.1) is

$$\Psi(\mathbf{t}, \mathbf{z}) = \mathbf{t}^A\mathbf{z}.$$

Matrix partial differentiation with respect to $\mathbf{t}$, and setting all $t_i = 1$ gives the infinitesmal generators:

$$\mathbf{v} = A\frac{\partial}{\partial \mathbf{z}}.$$

An interpretation of the toric variety is as characterising the orbits of the group, as discussed above. We have not formally proved the Buckingham theorem, but drawing on the above discussion it is give as Theorem 2.22 in [14].

5. Discussion

We have seen that the toric ideal method, via the Graver basis, is a fast way to compute all primitive invariants in dimensional analysis. There are three areas of further study which this suggests.

The first area arises from the possibility that different physical systems may yield different types of toric ideal or variety. The most important general class is *normal* toric varieties. Briefly such varieties are related to polyhedral cones and polyhedra with integer or rational generators. The standard approach is to take the such a cone σ and compute its Hilbert basis, which is a set of integer generators of the dual cone which gives all integer grid points in that cone. From this there is a natural toric ideal. But an open problem, it seems to the authors, is whether the rich theory of normal and polyhedra has a role in classical physics and engineering.

The second area would be the natural development from the last section. A discussion missing from in this paper is the way in which differentials are convert to derived quantities. For example velocity, which is $\frac{\partial y}{\partial t}$, for some length variable y and time t is awarded the derive quantity LT^{-1}. One way to keep the advantages of awarding derived quantities to differential terms, but keep the meaning of differentials is to use combinations of differential and polynomial operators. The algebraic environment which combines differential operators of this kind with polynomial algebras are differential algebras and in particular Weyl and Orr algebras. It would be useful to develop a type of generalization of dimensional analysis which combined differential algebras with the invariance touched on in the last section.

The third area is considered because the authors came to this work from the use of experimental design methods in engineering. It seems that having easy access to *all* primitive invariants should expand the scope of experimental design methods based on invariants, which is a small but established field see [11] [12]. This is mention briefly in (3.1). The authors hope to develop this idea.

References

1. Barnett, W., *Dimensions and economics: some problems*, Quartertly J. Austrian Economics **1** (2007), no. 95-104.
2. Bhaskar, R. and Nigan, A., *Qualitative physics using dimensional analysis*, Artifivcal Intelligence **45** (1990), 73–111.
3. Hershberger, R.E. Bolster, D. and Donelly, R. J., *Dynamic silimarity, the dimensionaless science*, Physices today (2011), 42–46.
4. Buckingham, E., *On physically similar systems: illustration of the use of dimensional analysis*, Phys, Rev **4** (1914), 345–376.
5. Chapman, S. C. and Watkins, N. C., *Avalanching systems under intermediate driving rate*, Plasma Phys. Control Fusion (2009), 1–9.
6. Rowlands, G Chapman, S. C. and JOURNAL = Physics of Plasmas YEAR = 2009 volume = 16 pages = 012303 Watkins, N. C., TITLE = Macroscopic for control parameter for avalanche models for bursty transport .
7. CoCoATeam, CoCoA*: a system for doi ng Computations in Commutative Algebra*, Available at `http://cocoa.dima.unige.it`.
8. Little, J. B. Cox, D. A. and Schenck, H.K., *Toric varieties*, 124 ed., Graduate Studies in Mathematics, American Mathematicval Society, Providence, Rhode Island, 2011.
9. D'Agostino, S., *A history of the ideas of theoretical physics*, Kluwer, Dordrecht, 2002.

10. Gibbings, J. C., *The systematic experiment*, Cambridge University Press, Cambridge, 1986.
11. ———, *Dimensional analysis*, Springer, London, 2011.
12. Grove, D. M. and Davis, T. P., *Engineering, quality and experimental design*, Longmans, London, 1992.
13. Jensen, A. N., *Gfan, a software system for groebner fans*, http://www.math.tu-berlin.de/ jensen/software/gfan/gfan.html (2012).
14. Olver, P. J., *Application of lie groups to differential equations*, Springer-Verlag, New York, 1986.
15. Smith, H., *An introduction to quantum mechanics*, World Scientific, New York, 1991.
16. Sturmfels, B., *Gröbner bases and convex polytopes*, 8 ed., University Lecture Series, American Mathematicval Society, Providence, Rhode Island, 1996.

Brunel University, Middlesex, UK
E-mail address: `Mark.Atherton@brunel.ac.uk`

Rolls-Royce plc, Derby, UKJ
E-mail address: `Ron.Bates@Rolls-Royce.com`

London School of Economics, London, UK
E-mail address: `h.wynn@lse.ac.uk`